Ph.D. Syllabus

Mathematical Sciences

Ph.D. MATHEMATICAL SCIENCE

MTH 601: Algebra I (4)

Monoids, Groups, group actions, Sylow's theorems, Finitely generated abelian groups, free groups

Rings and homomorphisms, Chinese remainder theorem, examples as polynomial ring and power series ring, rings of endomorphisms, Universal property of polynomial rings, Localization, Principal and factorial rings

Modules, quotient modules, direct product and direct sum of modules, Jordan-Hölder theorem, Free, Projective and Injective modules, Dual modules, Modules over PID

Category and Functor, Direct and Inverse limit

Polynomials in one and several variables, Gauss lemma, Irreducibility criterions, Power series ring, group of units in power series ring

Algebraic extensions, Algebraic closure, Splitting fields, Normal extensions, Separable and inseparable extensions, Finite fields

Galois extensions, examples, Galois correspondence theorems, solvability of equations

***Suggested Books*:**

- Algebra (3rd Edition), Serge Lang, Addison Welsey
- Basic Algebra, Jacobson, parts-I and II, Dover Publications Inc.; 2nd edition.
- Algebra, Birkhoff and McLane, Chelsea Publishing Co.
- A course in theory of groups, D.J.S. Robinson, Springer; 2nd ed.

MTH 602: Algebra II (4)

Extension of rings, integral extensions, going up and going down theorems, integral closure, integral galois extensions

Transcendental extension, transcendence basis, Noether normalization theorem, separable and regular extensions

Algebraic varieties, Hilbert's Nullstellensatz, Spec of a ring

Noetherian (and Artinian) rings and Modules

Matrices and linear maps, determinants, duality, bilinear and quadratic forms

Tensor product, basic properties, bimodules, Flat modules, extension of scalars, Algebras, Graded algebras, Tensor, symmetric and exterior algebras

***Suggested Books*:**

- S. Lang, Algebra, 3rd Edition, Addison Wesley.
- Jacobson, Basic Algebra I and II.
- Birkhoff and McLane, Algebra.
- Dummit and Foote, Abstract Algebra, 2nd Edition, Wiley.

MTH 603: Real Analysis (4)

Several variable calculus: A quick overview, the contraction mapping theorem, the inverse function theorem, the implicit function theorem.
Riemann integration in $\mathbf{R}^n$, n≥1.

Lebesgue measure and integration: Measures, measurable functions, integration of nonnegative and complex functions, modes of convergence, convergence theorems, product measure, Fubini's theorem, convolution, integration in polar coordinates.

Signed measures and differentiation, complex measures, total variation, absolute continuity, Fundamental theorem of calculus for Lebesgue integral, the Radon-Nikodym theorem and consequences.

$\mathbf{L}^p$ spaces, the Hölder and Minkowski inequalities, Jensen's inequality, completeness, the Riesz representation theorem, dual of $\mathbf{L}^p$ spaces.

***Suggested Books*:**

- G.B. Folland, Real analysis: Modern techniques and their applications, 2nd Edition, Wiley.
- W. Rudin, Principles of Mathematical Analysis, 3rd Edition, Tata McGraw-Hill.
- W. Rudin, Real and Complex Analysis, 3rd Edition, Tata McGraw-Hill.
- E.M. Stein and R. Shakarchi, Functional Analysis: Introduction to further topics in analysis, Princeton lectures in analysis.
- T. Tao, Analysis I and II, 2nd Edition, TRIM Series 37, 38, Hindustan Book Agency.

MTH 604: Complex Analysis II (4)

***Pre-requisites**:*
Required: MTH 303 Real Analysis I, MTH 407 Complex Analysis
Desirable: MTH 304 Metric Spaces and Topology, MTH 503 Functional Analysis

Review of elementary concepts: Complex differentiation, Cauchy-Riemann equations, holomorphicity, complex integration, Cauchy's theorem and Cauchy's integral formula, Taylor and Laurent series, residue theorem, definition of a meromorphic function.

Harmonic functions: definition and properties, Poisson integral formula, mean-value property, Schwarz reflection principle, Dirichlet problem

Maximum modulus principle: Maximum modulus theorem, Schwarz lemma, Phragmen-Lindelof theorem

Approximations by rational functions: Runge's theorem, Mittag-Leffler theorem

Conformal mappings: definition and examples, space of holomorphic functions, Montel's theorem, statement of Riemann mapping theorem

Entire functions, Infinite products, Weierstrass factorization theorem, little and big Picard Theorems, Gamma function

***Suggested Books*:**

Texts:

- Stein E.M. and Shakarchi R., *Complex Analysis (Princeton Lectures in Analysis Series, Vol. II)*, Princeton University Press, 2003
- Conway J.B., *Functions of One Complex Variable*, Springer-Verlag NY, 1978
- Rudin W., *Real and Complex Analysis*, McGraw-Hill, 2006
- Epstein B. and Hahn L-S., Classical Complex Analysis, Jones and Bartlett, 2011
- Ahlfors L., *Complex Analysis*, Lars Ahlfors, McGraw-Hill, 1979.

References:

- Carathodory C., *Theory of functions of a complex variable, Vol. I and II*, Chelsea Pub Co, NY 1954
- Remmert R., Classical topics in complex function theory, Springer 1997

MTH 605: Topology I (4)

General Topology: Connectedness, Compactness, Local Compactness, Paracompactness, Quotient Spaces, Topological Groups, and Baire Category Theorem.

Fundamental Gruops and Covering Spaces: Homotopy, Homotopy Equivalence and Deformation Retractions, Fundamental Group, Van Kampen Theorem, Deck Transformations, Group Actions, and Classification of covering spaces.

Cellular and Simplicial Complexes: Operations on Cell Complexes and Homotopy Extension Property. Simplicial Complexes - Barycentric Subdivision and Simplicial Approximation Theorem.

***Suggested Books*:**

- A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.
- J. R. Munkres, Topology, Second Edition, Prentice Hall, 2011.
- G. Bredon, Topology and Geometry, Springer International Edition, 2006.
- W. S. Massey, A Basic Course in Algebraic Topology, Springer International Edition, 2007.
- J. J. Rotman, An Introduction to Algebraic Topology, Springer, 1988.
- J. R. Munkres, Elements of Algebraic Topology, Westview Press, 1996.

MTH 606: Ordinary Differential Equations (4)

Pre-requisites*: MTH 201, MTH 303, MTH 403*

First-order equations

- Direction fields, approximate solutions , the fundamental inequality, uniqueness and existence theorems, solutions of equations containing parameters
- Comparison theorems

Systems of first-order equations

- Linear systems with constant coefficients: exponentials of linear operators, the fundamental theorem for linear systems, linear systems in the plane, canonical forms of linear operators on a complex vector space (S+N decomposition, nilpotent canonical forms, Jordan and real canonical forms), stability theory (saddle, spiral, and nodal points), phase portraits
- Linear equations of higher order: fundamental systems, Wronskian, reduction of order, non-homogeneous linear systems, Green's function
- Non-linear systems: the fundamental existence-uniqueness theorem, dependence on initial conditions and parameters, the maximal interval of existence, the flow defined by a differential equation, linearization, the Stable Manifold theorem, the Hartman-Grobman theorem, stability theory of equilibria (saddles, nodes, foci and centres), Liapunov functions, La-Salle's invariance principle, gradient systems

Poincare-Bendixson theory: Limit sets, local sections, theorem of Poincare-Bendixson, Poincare's index, orbital stability of limit cycles, index of simple singularities

***Suggested Books*:**

- G. Birkhoff & G. C. Rota, Ordinary differential equations, Paperback edition, John Wiley & Sons, 1989
- W. Hurewicz, Lectures on ordinary differential equations, Dover, New York, 1997
- Morris Hirsch and Stephen Smale, Differential Equations, Dynamical Systems, and Linear Algebra (Pure and Applied Mathematics (Academic Press), 1974
- P. Hartman, Ordinary Differential Equations, New York, Wiley, 1964

MTH 608: Introduction to Differentiable Manifolds and Lie Groups (4)

***Pre-requisites**: MTH 303 Real Analysis I, MTH 304 Metric Spaces and Topology, MTH 306 Ordinary Differential Equations, MTH 403 Real Analysis II*

Differentiable manifolds: definition and examples, differentiable functions, existence of partitions of unity, tangent vectors and tangent space at a point, tangent bundle, differential of a smooth map, inverse function theorem, implicit function theorem, immersions, submanifolds, submersions, Sard's theorem, Whitney embedding theorem

Vector fields: vector fields, statement of the existence theorem for ordinary differential equations, one parameter and local one-parameter groups acting on a manifold, the Lie derivative and the Lie algebra of vector fields, distributions and the Frobenius theorem

Lie groups: definition and examples, action of a Lie group on a manifold, definition of Lie algebra, the exponential map, Lie subgroups and closed subgroups, homogeneous manifolds: definition and examples

Tensor fields and differential forms: cotangent vectors and the cotangent space at a point, cotangent bundle, covector fields or 1-forms on a manifold, tensors on a vector space, tensor product, symmetric and alternating tensors, the exterior algebra, tensor fields and differential forms on a manifold, the exterior algebra on a manifold

Integration: orientation of a manifold, a quick review of Riemann integration in Euclidean spaces, differentiable simplex in a manifold, singular chains, integration of forms over singular chains in a manifold, manifolds with boundary, integration of n-forms over regular domains in an oriented manifold of dimension n, Stokes theorem, definition of de Rham cohomology of a manifold, statement of de Rham theorem, Poincare lemma

***Suggested Books*:**

Texts:

- J. Lee, *Introduction to smooth manifolds*, Springer, 2002
- W. Boothby, *An Introduction to differentiable manifolds and Riemannian geometry,* Academic Press, 2002
- F. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer, GTM 94, 1983
- M. Spivak, *A comprehensive introduction to differential geometry, Vol. 1*, Publish or Perish, 1999

References:

- G. de Rham, *Differentiable manifolds: forms, currents and harmonic forms*, Springer, 1984
- V. Guillemin and A. Pollack., *Differential topology,* AMS Chelsea, 2010
- J. Milnor, *Topology from the differentiable viewpoint,* Princeton University Press, 1997
- J. Munkres, *Analysis on manifolds,* Westview Press, 199
- 01254217
- C. Chevalley, *Theory of Lie groups,* Princeton University Press, 1999

- R. Abraham, J. Marsden, T. Ratiu, *Manifolds, tensor analysis, and applications*, Springer, 1988

MTH 609: Sturm-Liouville Theory (4)

Pre-requisites*: MTH 306 Ordinary Differential Equations, MTH 404 Measure and Integration*

Fourier Series: Fourier series of a periodic function, question of point-wise convergence of such a series, behavior of the Fourier series under the operation of differentiation and integration , sufficient conditions for uniform and absolute convergence of a Fourier series, Fourier series on intervals, examples of boundary value problems for the one dimensional heat and wave equations illustrating the use of Fourier series in solving them by separating variables, a brief discussion on Cesaro summability and Gibbs phenomenon

Orthogonal Expansions: A quick review of L2 spaces on an interval, convergence, completeness, orthonormal systems, Bessel's inequality, Parseval's identity, dominated convergence theorem
Sturm-Liouville Systems: linear differential operators, formal adjoint of a linear operator, Lagrange's identity, self-adjoint operators, regular and singular Sturm-Liouville systems, Sturm-Liouville series, Prufer substitution, Sturm comparison and oscillation theorems, eigenfunctions, Liouville normal form, distribution of eigenvalues, normalized eigenfunctions, Green's functions, completeness of eigenfunctions

Illustrative boundary value problems: A technique to solve inhomogeneous equations using Sturm-Liouville expansions, one dimensional heat and wave equations with inhomogeneous boundary conditions, one dimensional inhomogeneous heat and wave equations, mixed boundary conditions, Dirichlet problem in a rectangle and a polar coordinate rectangle

Maximum Principle and applications: maximum principle for linear, second-order, ordinary differential equations, generalized maximum principle for such equations, applications to initial and boundary value problems, the eigenvalue problem, an extension of the principle to non-linear equations

Orthogonal polynomials and their properties: Legendre polynomials, Legendre equation, Legendre functions and spherical harmonics, Hermite polynomials, Hermite functions, Hermite equation, Laguerre polynomials, Laguerre equation, zeros of orthogonal polynomials on an interval, and a recurrence relation satisfied by them

Bessel Functions: Bessel's equation, identities, asymptotics and zeros of Bessel functions

***Suggested Books*:**

Texts:

- Birkhoff, G & Rota G., *Ordinary Differential Equations*, John Wiley & Sons
- Folland, G., *Fourier Analysis & Its Applications,* AMS
- Protter, M. & Weinberger, H., *Maximum Principles in Differential Equations*, Springer

References:

- Brown, J. & Churchill, R., *Fourier Series and Boundary Value Problems*, McGraw-Hill
- Jackson, D., *Fourier Series and Orthogonal Polynomials*, Dover

MTH 610: Fourier Analysis on the Real Line (4)

Pre-requisites: *MTH 404 Measure and Integration, MTH 503 Functional Analysis: Normed linear spaces, completeness, Uniform boundedness principle, MTH 405 Partial Differential Equations: Basic knowledge of Laplacian, Heat and Wave equations*

The vibrating string, derivation and solution to the wave equation, The heat equation

Definition of Fourier series and Fourier coefficients, Uniqueness, Convolutions, good kernels, Cesaro/Abel means, Poisson Kernel and Dirichlet's problem in the unit disc

Mean-square convergence of Fourier Series, Riemann-Lebesgue Lemma, A continuous function with diverging Fourier Series

Applications of Fourier Series : The isoperimetric inequality, Weyl's equidistribution Theorem, A continuous nowhere-differentiable function, The heat equation on the circle

Schwartz space*, Distributions*, The Fourier transform on $\mathbf{R}$: Elementary theory and definition, Fourier inversion, Plancherel formula, Poisson summation formula, Paley-Weiner Theorem*, Heisenberg Uncertainty principle, Heat kernels, Poisson Kernels

(If time permits/possible project topic) Definition of Fourier transform on $\mathbf{R}^d$, Definition of X-ray transform in $\mathbf{R}^2$ and Radon transform in $\mathbf{R}^3$, Connection with Fourier Transform, Uniqueness

***Suggested Books*:**

Texts:

- E.M.Stein and R. Shakarchi, *Fourier Analysis: An Introduction, Princeton Univ Press*, 2003
- (For topics marked with a*) W. Rudin, *Functional Analysis,* 2nd Ed, Tata McGraw-Hill, 2006

References:

- J. Douandikoetxea*, Fourier Analysis* (Graduate Studies in Mathematics), AMS, 2000
- L. Grafakos, *Classical Fourier Analysis* (Graduate Texts in Mathematics), 2nd Ed, Springer, 2008

MTH 612: Non-commutative Algebra (4)

***Pre-requisites**: MTH 301, MTH 302, MTH 401*

Matrix Rings and PLIDs, Tensor Products of Matrix Algebras, Ring constructions using Regular Representation

Basic notions for Noncommutative Rings, Structure of Hom(M,N), Semisimple Modules & Rings, the Wedderburn Structure Theorem, Simple Rings, Rings with Involution

The Jacobson Radical and its properties, Primitive Rings and Ideals, Hopkins-Levitzki Theorem, Nakayama's Lemma , Radical of a Module, Local Rings, Chevalley-Jacobson Theorem, Kolchin's Theorem, Clifford Algebras.

Prime and Semiprime rings, Rings of Fractions and Goldie's Theorems, Rings with ACC (ideals), Tensor Algebras, Algebras over large Fields, Deformations and Quantum Algebras.

Hereditary Rings and their Modules, Division rings.

Central Simple Algebras, Cyclic Algebras, Symbol Algebras, Crossed Products, the Brauer Group, the functor Br, the Skolem-Noether Theorem, the centralizer Theorem, calculation of Brauer group of commutative rings.

***Suggested Books*:**

- L. Rowen, Graduate algebra: noncommutative view, Graduate Studies in Mathematics, 91.
- B. Farb, R. Dennis, Noncommutative algebra, GTM, Springer-Verlag.
- T. Y. Lam, A first course in noncommutative rings, GTM, Springer.
- J. Golan and T. Head, Modules and the structure of rings: A primer, Pure and applied mathematics

MTH 613: Introduction to Riemannian Geometry (4)

Pre-requisites*: MTH 405 and MTH 508*

Review of differentiable manifolds: vector bundles, tensors, vector fields, differential forms, Lie groups

Riemannian metrics. Definition, examples, existence theorem; model spaces of Riemannian geometry

Connections: connections on a vector bundle, linear connections, covariant derivative, parallel transport, geodesics

Riemannian connections and geodesics: torsion tensor, Fundamental Theorem of Riemannian Geometry, geodesics of the model spaces, exponential map, convex neighborhoods, Riemannian distance function, first variation formula, Gauss' lemma, geodesics as locally minimizing curves; completeness, statement of Hopf-Rinow Theorem

Curvature: Riemann Curvature Tensor, Bianchi identity, scalar, sectional and Ricci curvatures

Jacobi Fields: Jacobi equation, conjugate points, second variation formula, spaces of constant curvature (if time permits)

Curvature and topology: Gauss-Bonnet Theorem, Bonnet-Myers Theorem, Cartan-Hadamard Theorem

Suggested Books**:**

Texts:

- J. M. Lee. Riemannian Manifolds, An introduction to Curvature. Graduate Texts in Mathematics. Springer (1997).
- M. P. do Carmo. Riemannian Geometry. Birkhauser (1991).
- S. Gallot, D. Hulin, J. Lafontaine. Riemannian Geometry. Springer (2004).

References:

- I. Chavel. Riemannian geometry, a modern introduction. Cambridge University Press (2006)
- S. Kobayashi, K. Nomizu. Foundations of differential geometry, vol. -I, Wiley Interscience Publication (1996).

MTH 614: Functional Analysis (4)

Pre-requisites (Desirable)*: MTH 404: Measure and Integration*

***Learning Objectives*:**

Functional analysis is the branch of mathematics concerned with the study of spaces of functions. This course is intended to introduce the student to the basic concepts and theorems of functional analysis with special emphasis on Hilbert and Banach Space Theory. This gives the basics for more advanced studies in modern Functional Analysis, in particular in Operator Algebra Theory and Banach Space Theory.

***Course Contents*:**

- Normed Linear spaces, Bounded Linear Operators, Banach Spaces, Finite dimensional spaces, Quotient Spaces
- Hilbert spaces, Riesz Representation Theorem, Orthonormal sets, Bessel's Inequality, Parseval's Identity, Fourier Series
- Dual Spaces, Dual of L^p spaces , Hahn-Banach Extension Theorem, Applications
- Open Mapping Theorem, Closed Graph Theorem, Uniform Boundedness Principle
- Weak and Weak-* topologies, Hahn-Banach Separation Theorem, Alaoglu's Theorem, Reflexivity
- Compact Operators, Adjoint of an operator, Spectral theorem for Compact Self-Adjoint operators
- (If time permits) Banach Algebras, Ideals and Quotients, Gelfand-Mazur Theorem, Fredholm Alternative, Fredholm Operators, Atkinson's theorem

***Suggested Books*:**

- J.B. Conway, A Course in Functional Analysis, 2nd Ed., (Springer-Verlag, 1990)
- S. Kesavan, Functional Analysis, TRIM 52, Hindustan Book Agency
- B.V. Limaye, Functional Analysis, 2nd Ed., (New Age International, 1996)
- Martin Schechter, Principles of Functional Analysis, 2nd Ed., Graduate Studies in Mathematics, AMS
- P.D Lax, Functional Analysis, (Wiley, 2002)
- W. Rudin, Functional Analysis, 2nd Ed., (Tata McGraw-Hill, 2006)

MTH 615: Operator Theory and Operator Algebras (4)

***Pre-requisites**: MTH 503 Functional Analysis*

Course Contents:

Banach Algebras, Ideals, Quotients, homomorphisms, Unitization

Invertible Elements, Spectrum, Gelfand-Mazur Theorem, Spectral Radius Formula

Commutative Banach Algebras, The Gelfand Transform, Applications to Fourier Transforms, Weiner's Theorem, Stone-Weierstrass Theorem

Compact and Fredholm Operators, Atkinson's Theorem, Index Theory

C* algebras, uniqueness of the norm, Commutative C* algebras, Gelfand-Naimark theorem, Spectral Mapping theorem

Functional Calculus, Positive Operators, Polar Decomposition

Weak and Strong Operator Topologies, Von Neumann Algebras, Double Commutant Theorem

Spectral measure, Spectral Theorem for Normal Operators, Borel Functional Calculus

Multiplicity Theory, Abelian Von Neumann Algebras, Classification of normal operators upto unitary equivalence

***Suggested Books*:**

- G. J. Murphy, *C* Algebras and Operator Theory* (Academic Press Inc, 1990)
- J. B. Conway, *A Course in Functional Analysis* (2^{nd} Ed) (Springer, 1990)
- R. G. Douglas, *Banach Algebra Techniques in Operator Theory* (2^{nd} Ed) (Springer, 1998)
- K. R. Davidson, *C* Algebras by Example* (Fields Institute Monograph, AMS 1996)
- R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras - Vol. I* (Academic Press Inc, 1983)
- W. A. Arveson, *A Short Course in Spectral Theory* (Springer 2002)

MTH 616: Topology II (4)

Pre-requisites (Desirable)*: MTH 507 or MTH 605 and MTH 302 or MTH 601*

Learning Objectives**:**

This is an advanced course in Topology.

Course Contents**:**

Simplicial Homology: Simplicial Complexes, Barycentric Subdivision, and Simplicial Homology with examples

Singular and Cellular Homology: Definition with examples, Homotopy Invariance, Exact Sequence of Relative Homology, Excision, Mayer-Vietoris Sequence, Degree of Maps, and Cellular Homology, Jordan-Brouwer Separation Theorem, Invariance of domain and dimension, Borsuk-Ulam Theorem, Lefschetz-Hopf Fixed Point Theorem, Axioms for homology, Fundamental group and homology, and Simplicial Approximation Theorem

Cohomology: Universal Coefficient Theorem, Künneth Formula, Cup Product and the Cohomology Ring, Cap Product, Orientations on Manifolds, and Poincaré Duality

Higher Homotopy Groups: Definition with examples, Aspherical Spaces, Relative Homotopy Groups, Long Exact Sequence of a triple, n-connected spaces, and Whitehead's Theorem

Suggested Books**:**

- A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.
- E. H. Spanier, Algebraic Topology, Springer, 1994.
- J. R. Munkres, Elements of Algebraic Topology, Westview Press, 1996.
- J. J. Rotman, An Introduction to Algebraic Topology, Springer, 1988.
- M. J. Greenberg & J. R. Harper, Algebraic Topology: A First Course, Perseus Books Publishing, 1981.
- W. S. Massey, A Basic Course in Algebraic Topology, Springer International Edition, 2007.
- G. Bredon, Topology and Geometry, Springer International Edition.

MTH 617: Introduction to Algebraic Geometry (4)

***Learning Objectives*:**

This course aims to provide an introduction to some of the basic objects and techniques and objects of algebraic geometry with minimal prerequisites. The main emphasis is on geometrical ideas and so most of the treatment will be over algebraically closed fields. Results from commutative algebra will introduced and proved as required and so no prior experience with commutative algebra will be assumed. After introducing the basic objects and techniques, they will be illustrated by application to the theory of algebraic curves.

***Course Contents*:**

Closed subsets of affine space, coordinate rings, correspondence between ideals and closed subsets, affine varieties, regular maps, rational functions, Hilbert's nullstellensatz

Projective and quasi-projective varieties, regular and rational functions on projective varieties, products and maps of quasi-projective varieties

Dimension of varieties, examples and applications

Local ring of a point, tangent and cotangent space, local parameters, non-singular points and non-singular varieties

Birational maps, blowups, disingularization of curves

Intersection numbers for plane curves, divisors on curves, Bezout's theorem, Riemann-Roch theorem for curves, Residue theorem, Riemann-Hurwitz formula

***Suggested Books*:**

- W. Fulton, Algebraic curves: An introduction to algebraic geometry, 2008 ed. (available online).
- R. Shafarevich, Basic Algebraic Geometry, Vol. 1, Third Edition, Springer, Heidelberg, 2013.
- S. Abhyankar, Algebraic geometry for scientists and engineers, Mathematical Surveys and Monographs 35, American Mathematical Society, 1990.
- K. Smith et al, An invitation to algebraic geometry, Springer, 2004.